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THE BEHAVIOR OF A FLAT ELLIPTICAL CRACK IN AN ANISOTROPIC ELASTIC BODY

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Abstract—A method devolving upon the computation of certain influence coefficients is herein presented for determining the material displacements and stress in the vicinity of the edge of an elliptic crack within an arbitrarily anisotropic elastic body. In particular, compact line integral expressions for the stress intensity factors about the circumference of the crack and for the magnitude of the crack face displacement are derived. In all cases, the elastic body is assumed subject to uniform stress states far from the crack. Numerical results for a special example are also shown.

INTRODUCTION

Almost all of the extant treatments of elastic inclusions and cracks draw upon the concepts developed by Eshelby in his now classic paper[1], which deals mostly with inclusions and inhomogeneities in isotropic bodies. Walpole[2] has drawn upon this formalism to generate an iteration scheme for obtaining the strain of an inclusion in an anisotropic body to any desired degree of accuracy, but his end results are not given in closed form. Willis[3] specifically addressed the problem of the stress field around an elliptical crack in a linear anisotropic elastic medium. His work is not directly applicable to cases of transverse isotropy owing to degeneracies which occur in his solutions when applied to transverse isotropy. Mura and Lin[4] restricted their analysis to cracks in cubic media, and the work of Kassir and Sih[5] on cracks in transversely isotropic bodies contains some serious errors. (One implication of this latter research is that the crack faces undergo no displacement for samples subject to tension perpendicular to the crack plane, a physical impossibility. In addition, stress intensity factors calculated from their results may be imaginary.) The tools of dislocation theory[6] have also been brought to bear on this problem, but these results require calculations of multiple integrals, a procedure thought not to be as easy to implement as that followed below. Laws[13] develops the theory of interfacial discontinuities to handle arbitrary cavities in anisotropic media, of which flat cracks are a special case.

The method of anisotropic crack analysis adopted in the present study is an extension of the method developed by Irwin[7] for analyzing flat cracks in isotropic bodies.‡

ANALYSIS

The planform of the crack is elliptical and characterized by semi-axes a and b, $a \geq b$. The *planform aspect ratio* is $\gamma = b/a$, $\gamma \leq 1$. A crack coordinate system $(x_1, x_2, x_3) = (x, y, z)$ is chosen so that the x-axis coincides with major axis and the z-axis is normal to the plane of the crack. With respect to this coordinate system, the body is characterized by elastic constants M_{ijkl}. Far from the crack, uniform stresses σ_{ij}^{∞} are applied.

It is helpful to regard the crack as the limiting member of a certain family of oblate ellipsoidal cavities. These cavities are characterized by semi-axes a, b, c, where c varies among the various members of this family. The crack is the limit member of this family as the thickness aspect ratio $\alpha = c/b \to 0$. Consideration of these cavities will allow important conclusions to be drawn about the nature of the crack face displacements under the action of external stresses.

Eshelby[1] has proved an important theorem about the strains of an ellipsoidal inhomogeneity in an anisotropic body subject to uniform far stress: the inclusion strains are

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‡I am indebted to J. R. Rice for a crucial suggestion making this extension possible.

uniform. This theorem applies to ellipsoidal cavities, which are inhomogeneities with vanishingly small stiffnesses, and further implies that the cavity interface displacements U_i will be linear functions of the coordinates x, y, z:

$$U_i = A_i x + B_i y + C_i z. \tag{1}$$

For this expression to remain meaningful in the limit of $\alpha \to 0$, it is useful to introduce the coefficient β_i defined by

$$C_i = \beta_i \frac{\sqrt{(ab)}}{c} \cdot \frac{\sigma}{E} \tag{2}$$

when σ is a typical stress, and E one of the moduli of the body. For vanishing α, the inclusion strains ϵ_{11}^I, ϵ_{12}^I, ϵ_{22}^I remain bounded and are therefore negligible in comparison with the remaining strain components ϵ_{13}^I, ϵ_{23}^I, ϵ_{33}^I:

$$\begin{aligned} \epsilon_{31}^I &= \frac{\sigma}{E} \cdot \frac{\beta_1}{2\alpha\sqrt{\gamma}} \\ \epsilon_{32}^I &= \frac{\sigma}{E} \cdot \frac{\beta_2}{2\alpha\sqrt{\gamma}} \\ \epsilon_{33}^I &= \frac{\sigma}{E} \cdot \frac{\beta_3}{\alpha\sqrt{\gamma}}. \end{aligned} \tag{3}$$

From (1)–(3), it therefore follows that for small α *and in particular for cracks*, the upper crack face displacements associated with the singular components are

$$U_i = \frac{\sigma}{E} \beta_i (ab)^{1/2} \sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)} \tag{4}$$

and that

$$\frac{E}{\sigma} \lim_{\alpha \to 0} [\alpha \epsilon_{ij}^I] = \frac{1}{2\sqrt{\gamma}} \beta_p (\delta_{ip}\delta_{j3} + \delta_{jp}\delta_{i3}). \tag{5}$$

The β_i will be called the *crack displacement magnitudes*.

The means for determining the β_i lie within a set of influence coefficients C_{jk}, whose existence is implied by the linearity of the problem. These coefficients connect the displacement magnitudes with the applied stress:

$$\sigma_{3j}^\infty = \sigma C_{jk} \beta_k \qquad \sigma \beta_j = C_{jk}^{-1} \sigma_{3k}^\infty. \tag{6}$$

The stress components other than σ_{3k}^∞ have no influence on the β_i. The bulk of this section will be devoted to determining the matrix (C_{jk}). The matrix (C_{jk}) is symmetric, as is shown in Appendix 1.

Associated with the presence of the crack is a defect ξ in the potential energy of the body. Expressions for the matrix (C_{jk}) are obtained by computing ξ in two different ways and equating the different, but equivalent, expressions.

First, note that ξ must equal the work done by the applied stress acting (gradually) through the displacement of each crack face. Thus

$$\xi = 2 \cdot \frac{1}{2} \iint_{\substack{\text{area}\\ \text{of the}\\ \text{crack}}} \sigma_{3k}^\infty U_k \, dx\, dy = \frac{\sigma^3}{E} C_{jk} \beta_j \beta_k \cdot \frac{2}{3} \pi (ab)^{3/2}. \tag{7}$$

A second expression for ξ is developed via exploitation of the so-called M surface integral

$$M = \iint_S \left\{ W\mathbf{x} \cdot \mathbf{n} - [(\mathbf{x} \cdot \nabla)\mathbf{U}]\mathbf{T} - \frac{1}{2} \mathbf{T} \cdot \mathbf{U} \right\} dS \tag{8}$$

which has the same value for all surfaces S that completely enclose the crack[8]. Here, $\mathbf{x}$ is the position vector, $\mathbf{U}$ is the elastic displacement, $\mathbf{T}$ is the surface traction on S, $\mathbf{n}$ is the unit outward normal to S, W is the strain energy density, and ∇ is the gradient operator. Budiansky and Rice[8] have interpreted this integral in terms of an energy release rate associated with the self-similar growth of the crack in which each point of the crack edge Γ recedes radially from the origin at a rate proportional to its distance therefrom. Specifically

$$M = a\frac{\partial \xi}{\partial a}. \tag{9}$$

Furthermore, Budiansky and O'Connell[9] show that

$$M = \oint_{\Gamma} \rho(s) \lim_{\delta \to 0} J(s, \delta)\, \mathrm{d}s. \tag{10}$$

(Although this expression is valid for cracks of arbitrary planform, it will be exclusively applied to elliptical cracks in this study.) Here, $\rho(s)$ is the perpendicular distance from the origin to the tangent line to Γ at s, and $J(s, \delta)$ is precisely Rice's well-known J-integral of two-dimensional fracture mechanics, subject to the understanding that $J(s)$ is to be evaluated in the plane $P(s)$ which is normal to the crack plane (x–y plane) at s (see Fig. 1). The path C in $P(s)$ on which J is to be evaluated is a circle, centered at s, of radius δ taken in the limit as $\delta \to 0$. In this limit, conditions of plane strain and antiplane shear are approached in $P(s)$[9].

The purpose of the analysis to be presented in the following paragraphs is to resolve further the analytic structure of $J(s, \delta)$ to permit an explicit computation of M. Begin by parameterizing the points $(\bar{x}, \bar{y})$ on the edge of the crack by means of the parameter ϕ:

$$(\bar{x}, \bar{y}) = (a \cos \phi, b \sin \phi). \tag{11}$$

Then

$$\rho(s)\, \mathrm{d}s = ab\, \mathrm{d}\phi \tag{12}$$

Ref. [9].

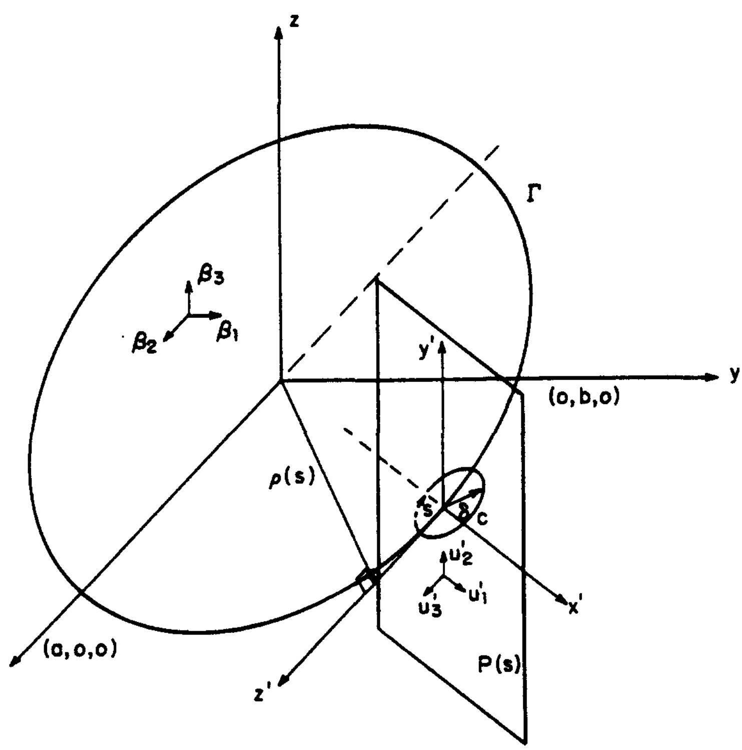


Fig. 1. Crack based coordinates.

At each point s on the edge of the crack, there is a right-handed edge coordinate system $(x'_1, x'_2, x'_3) = (x', y', z')$ oriented so that the x'-axis is normal to the edge, the z'-axis is tangent to it, and $z = y'$. The intersection of the flat crack and the x'–y' plane is a two-dimensional crack whose near tip stresses and displacements are amenable to the 2-D analysis of Refs. [10, 11].

Let the crack tip displacements in the x'–y' plane be U'_i. Asymptotic forms for these displacements can be derived by considering a point (x, y) in the crack coordinate system on the upper face of the crack near a point $(\bar{x}, \bar{y})$ on the edge in the same $(x'$–$y')$ plane. If ζ is the distance between the two, then

$$x = \bar{x} - \zeta n_x \qquad y = \bar{y} - \zeta n_y \tag{13}$$

where (n_x, n_y) are the components of the unit normal to Γ passing through (x, y). Substitution of eqn (13) into eqn (4) and transforming to the edge coordinate system gives

$$\begin{aligned} U'_1 &= \frac{\sigma}{E}(\gamma \cos\phi\beta_1 + \sin\phi\beta_2)(\gamma^2\cos^2\phi + \sin^2\phi)^{-1/4}\sqrt{(2a\zeta)} \\ U'_3 &= \frac{\sigma}{E}[-\sin\phi\beta_1 + \gamma\cos\phi\beta_2)(\gamma^2\cos^2\phi + \sin^2\phi)^{-1/4}\sqrt{(2a\zeta)} \\ U'_2 &= \frac{\sigma}{E}\beta_3(\sin^2\phi + \gamma^2\cos^2\phi)^{1/4}\sqrt{(2a\zeta)}. \end{aligned} \tag{14}$$

Letting

$$(R_{jk}) = (\sin^2\phi + \gamma^2\cos^2\phi)^{-1/4}\begin{pmatrix} \gamma\cos\phi & 0 & \sin\phi \\ 0 & (\sin^2\phi + \gamma^2\cos^2\phi)^{1/2} & 0 \\ -\sin\phi & 0 & \gamma\cos\phi \end{pmatrix} \tag{15}$$

and

$$\beta'_j = R_{jk}\beta_k \tag{16}$$

permits the following compact expression for the U'_i to be written

$$U'_i = \frac{\sigma}{E}\beta'_i\sqrt{(2a\zeta)}. \tag{17}$$

Next, consider the stress singularities and their associated stress intensity factors $K_{\rm I}$, $K_{\rm II}$, $K_{\rm III}$ at the tip. In order to continue using subscript notation and the summation convention, define k_i by

$$K_{\rm II} = \frac{\sigma}{E}k_1 \qquad K_{\rm I} = \frac{\sigma}{E}k_2 \qquad K_{\rm III} = \frac{\sigma}{E}k_3 \tag{18}$$

which are related to the limits

$$\frac{\sigma}{E}\begin{Bmatrix} k_1 \\ k_2 \\ k_3 \end{Bmatrix} = \lim_{\substack{\zeta\to 0 \\ y'=0}} \sqrt{(2\pi\zeta)}\begin{Bmatrix} \sigma'_{21} \\ \sigma'_{22} \\ \sigma'_{23} \end{Bmatrix} \tag{19}$$

where the σ'_{2i} are the components of σ_{jk} referred to the x'_i system.

In Appendix 2, an expression for the k_i is derived in terms of the β'_i (eqn 6 of Appendix 2):

$$k_j = -\sqrt{\pi}\, Q_{jk}\beta'_k. \tag{20}$$

Here, $Q_{ij} = Q_{ji}$ is a set of six real constants which are (known) functions of the elastic constants M'_{ijkl}; i.e. the M_{ijkl} referred to the edge coordinate system. The symmetry of Q_{ij} is shown in Appendix 2. In terms of this matrix, the J-integral is given by eqn (8) of Appendix 2:

$$\lim_{\delta\to 0} J(\phi,\delta) = \frac{-\sigma^2 a}{E}\frac{\pi}{2} Q_{jk}\beta'_j\beta'_k. \tag{21}$$

Substitution of this into (10) yields

$$\frac{a\partial\xi}{\partial a} = -\frac{\pi}{2}a^3\frac{\gamma\sigma^2}{E}\left[\int_\Gamma Q_{jk}\beta'_j\beta'_k\,\mathrm{d}\phi\right]. \tag{22}$$

Integrating, using (16), yields the promised second expression for the potential energy defect ξ

$$\xi = \frac{-\pi}{6}a^3\gamma\cdot\frac{\sigma^2}{E}\left[\int_\Gamma Q_{ij}R_{ik}R_{jl}\,\mathrm{d}\phi\right]\beta_k\beta_l. \tag{23}$$

Since the β's are arbitrary and (Q_{ij}) is symmetric, comparison of the two expressions for ξ, eqn (7) and (23), yields

$$C_{ij} = \frac{-1}{4\sqrt{\gamma}}\int_\Gamma Q_{kl}R_{ki}R_{lj}\,\mathrm{d}\phi, \tag{24}$$

valid for circular or elliptic cracks.

The crack displacements are now unambiguously determined by the applied stress state via (24) and (6). Furthermore, the stress intensity factors k_i at each point along the crack edge are known:

$$k_i = -\sqrt{\pi}\, Q_{ij}R_{jk}C^{-1}_{kl}\sigma^{\infty}_{l3}. \tag{25}$$

In all but the simplest of cases, the explicit calculations of the matrices Q_{ij}, C_{ij}, etc. require considerable assistance from a high speed digital computer.

TRANSVERSE ISOTROPY

One type of symmetry is simple enough so that the crack displacements, etc. can be explicitly calculated by hand without being too trivial to be of interest. This is the case of an elliptical crack parallel to the plane of isotropy of a transversely isotropic body. The major ellipse axis is parallel to the x-axis. In keeping with our earlier convention that the plane of isotropy coincides with the x–y plane of our laboratory frame, the matrix (M_{ij}) has the following form:

$$\underset{\sim}{M} = \frac{1}{E}\begin{pmatrix} 1 & -\nu_1 & -\nu_2 & & & \\ -\nu_1 & 1 & -\nu_2 & & & \\ -\nu_2 & -\nu_2 & \frac{1}{H} & & & \\ & & & \frac{2(1+\nu_1)}{\Gamma} & 0 & 0 \\ & & & 0 & \frac{2(1+\nu_1)}{\Gamma} & 0 \\ & & & 0 & 0 & 2(1+\nu_1) \end{pmatrix} \tag{26}$$

The five independent elastic constants are E, E_2, ν_1, ν_2, G_2; it was convenient to let

$$G_1 = \frac{E}{2(1+\nu_1)}, \qquad H = \frac{E_2}{E}, \qquad \Gamma = \frac{G_2}{G_1}.$$

For such a configuration, the components C_{ij} involve the following combinations of elastic constants

$$S = \sqrt{\left(\frac{1-\nu_1^2}{2}\right)}\sqrt{\left\{(1+\nu_1)\left(\frac{1}{\Gamma}-\nu_2\right)+\sqrt{\left[(1-\nu_1^2)\frac{1}{H}-\nu_2^2\right]}\right\}} \tag{27}$$

$$T = \frac{1+\nu_1}{\sqrt{\Gamma}} \tag{28}$$

$$\begin{aligned} R_1 &= \left\{E(k)\left[k^2-\left(1-\frac{S}{T}\right)\right]+\gamma^2K(k)\left(1-\frac{S}{T}\right)\right\}\Big/k^2 \\ R_2 &= \left\{E(k)\left[\left(1-\frac{S}{T}\right)\gamma^2+k^2\right]-\gamma^2K(k)\left(1-\frac{S}{T}\right)\right\}\Big/k^2. \end{aligned} \tag{29}$$

$K(k)$, $E(k)$ are, respectively, the complete elliptic integrals of the first and second kinds of the argument $k = \sqrt{(1-\gamma^2)}$. Then

$$\begin{aligned} C_{ij}^{-1} &= 0, \qquad i \neq j \\ C_{11}^{-1} &= 2S\sqrt{\gamma}/R_1 \\ C_{22}^{-1} &= 2S\sqrt{\gamma}/R_2 \\ C_{33}^{-1} &= 2S\sqrt{\gamma}/E(k)\cdot\sqrt{\left(\frac{(1/H)-\nu_2^2}{1-\nu_1^2}\right)}. \end{aligned} \tag{30}$$

In the limit of $b \to a$ (a circular flaw), the limiting forms of R_1, R_2, $E(k)$ need to be used:

$$\begin{aligned} E(0) &= \pi/2 \\ R_1 = R_2 &= \frac{\pi}{4}\left[1+\frac{S}{T}\right]. \end{aligned} \tag{31}$$

In the limit of $b \to 0$ (long ribbon-like filaments), the following limits are used:

$$\begin{aligned} E(1) &= 1 \\ R_1 &= \frac{S}{T} \\ R_2 &= 1. \end{aligned} \tag{32}$$

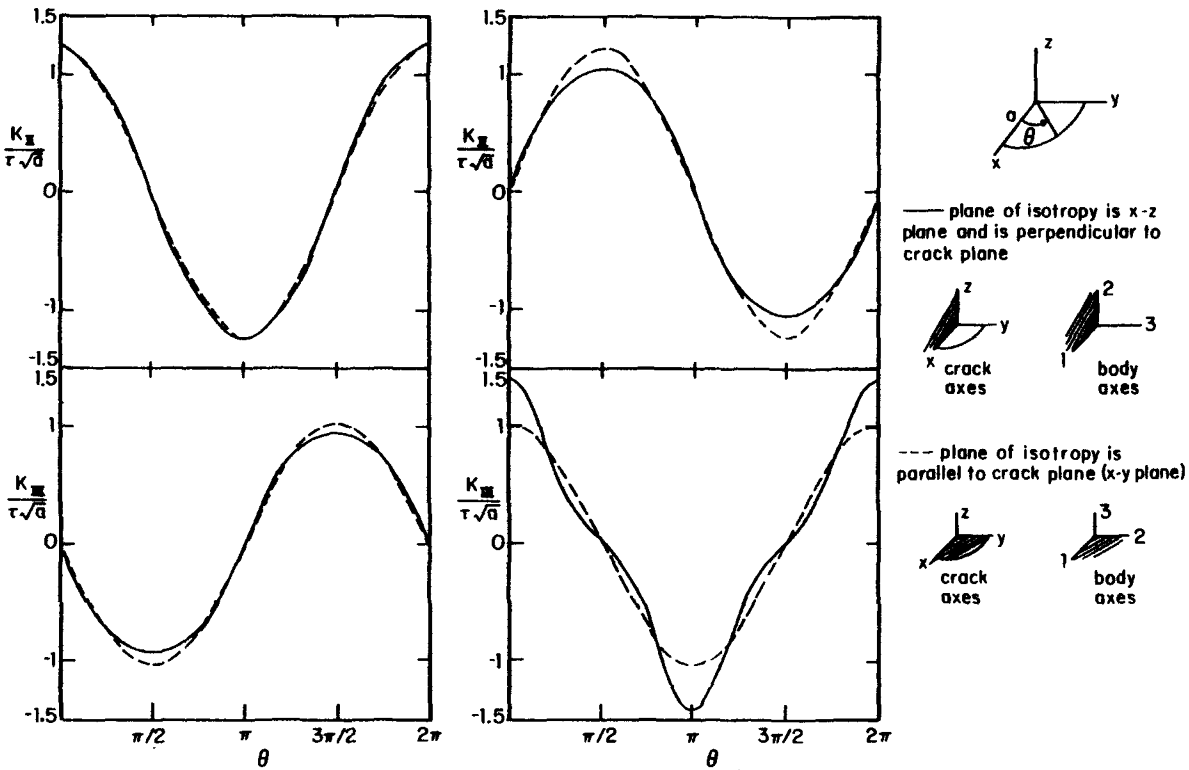
If a test stress $\sigma_{zz}^{\infty} = \sigma$ is applied, then

$$\frac{K_{\rm I}}{\sigma\sqrt{a}} = \frac{\sqrt{(\pi\gamma)}}{E(k)}(\sin^2\phi+\gamma^2\cos^2\phi)^{1/4}. \tag{33}$$

If a test stress of $\tau_{xz}^{\infty} = \tau$ is applied, then

$$\frac{K_{\rm II}}{\tau\sqrt{a}} = \frac{\sqrt{\pi}\,\gamma^{3/2}\cos\phi}{R_1(\sin^2\phi+\gamma^2\cos^2\phi)^{1/4}} \tag{34}$$

$$\frac{K_{\rm III}}{\tau\sqrt{a}} = \frac{-\sqrt{\pi}\,\gamma^{1/2}\sin\phi}{R_1(\sin^2\phi+\gamma^2\cos^2\phi)^{1/4}}\cdot\frac{S}{T}. \tag{35}$$



A. A test stress of $\tau = \tau_{xy}^{\infty}$ is applied

B. A test stress of $\tau = \tau_{yz}^{\infty}$ is applied

Fig. 2. Stress intensity factors, for a circular crack (radius a) in transversely isotropic body, shear deformation. The body is characterized by moduli $E_2/E = 2$, $G_2/E = 1$, $\nu_1 = 0$, $\nu_2 = 4$. Calculations were made for the plane of isotropy both parallel (dashed line) and perpendicular (solid curve) to the crack plane.

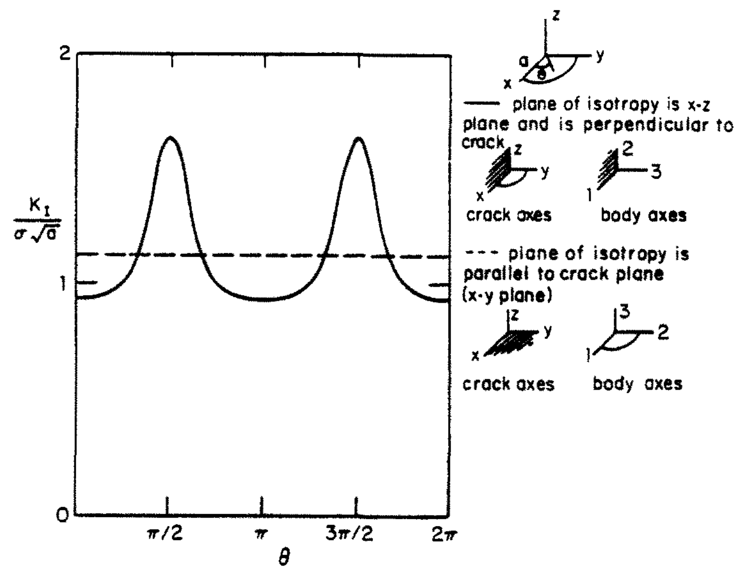


Fig. 3. Stress intensity factors for a circular crack in a transversely isotropic body, mode I deformation. A test stress of $\sigma = \sigma^{\infty}_{zz}$ is applied. The body is characterized by moduli $E_2/E = 2$, $G_2/E = 1$, $\nu_1 = 0$, $\nu_2 = 0.4$. Calculations are made for the plane of isotropy both perpendicular (solid curves) and parallel (dashed curve) to the plane of isotropy.

If a test stress of $\tau^{\infty}_{zy} = \tau$ is applied, then

$$\frac{K_{II}}{\tau\sqrt{a}} = \frac{\sqrt{\pi}\,\gamma^{1/2}\sin\phi}{R_2(\sin^2\phi + \gamma^2\cos^2\phi)^{1/4}} \tag{36}$$

$$\frac{K_{III}}{\tau\sqrt{a}} = \frac{\sqrt{\pi}\,\gamma^{3/2}\cos\phi}{R_2(\sin^2\phi + \gamma^2\cos^2\phi)^{1/4}}. \tag{37}$$

Values of the K's as functions of polar angle θ are indicated by the dotted line curves of Figs. 2 and 3, and for a particular solid characterized by $E_2/E = 2$, $\nu_1 = 0$, $\nu_2 = 0.4$, $G_2/E = 1$.

A similar calculation for the case of a circular crack embedded in the same solid when the plane of isotropy is now perpendicular to the plane of the crack can also be performed. This calculation must be done numerically, and these results are given by the solid curves of Figs. 2 and 3.† These figures indicate that the stress intensity factors for the two cases can be quite different. This difference is most pronounced for mode I deformation (Fig. 3). When the crack plane is parallel to the plane of isotropy, the stress intensity factor K_I is independent of the position along the crack edge, strongly contrasting with the considerable dependence of K_I with position when the crack plane is perpendicular to the plane of isotropy. Note, however, that when the applied shear is τ^{∞}_{13}, the difference between the two distributions for K_{II} and K_{III} is minimal, being at most about 10%.

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†Essentially this same calculation yields the magnitudes of the crack face displacement. F. Ghahremani has developed a program for predicting the displacements of arbitrary ellipsoidal cavities in anisotropic bodies[12]. An extrapolation from their predictions for very shallow oblate spheroids to cracks is in excellent agreement with the present calculations. These extrapolations agreed with the present calculations for circular cracks both parallel and perpendicular to the plane of isotropy.

Equations (34)–(37) are in disagreement with the stress-intensity factors given by Kassir and Sih[5] for cracks parallel to the plane of isotropy in transversely isotropic bodies.

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APPENDIX 1

Proof that the influence coefficients (C_{ij}) *are symmetric*

The symmetry of (C_{jk}), the matrix defined by eqn (6) of the main text, is demonstrated via an application of the Betti–Rayleigh Theorem. Let σ^1_{3j}, σ^2_{3j} give rise to displacements characterized by β_j^1, β_j^2, respectively. It is easy to see that β_j^1, β_j^2, also arise when tractions T_j^1, T_j^2 are applied to the surface of the crack, where $T_j^1 = +\sigma^1_{j3}\eta_3$ and $T_j^2 = +\sigma^2_{j3}\eta_3$ and there is no stress far from the crack. Let ξ_{12} be the work done by the external forces (1) acting through displacements (2):

$$\xi_{12} = +\int_A T_j^1\beta_j^2\frac{\sigma}{E}\sqrt{(ab)}\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}\,dA$$

where A is the surface of the crack. From eqn (6), this becomes

$$\xi_{12} = \frac{2\pi}{3}\frac{\sigma^2}{E}\sqrt{(ab)}abC_{jk}\beta_j^2\beta_k^1.$$

Similarly

$$\xi_{21} = \frac{2\pi}{3}\frac{\sigma^2}{E}\sqrt{(ab)}abC_{jk}\beta_j^1\beta_k^2.$$

But $\xi_{12} = \xi_{21}$, so that

$$C_{jk}\beta_j^2\beta_k^1 = C_{jk}\beta_j^1\beta_k^2 = C_{kj}\beta_k^1\beta_j^2$$

$$(C_{jk} - C_{kj})\beta_j^2\beta_k^1 = 0.$$

This equation holds for arbitrary β_j^1, β_j^2, so that

$$C_{jk} = C_{kj}$$

as asserted.

APPENDIX 2

The application of 2-D theory to flat elliptic crack analysis

The results for two-dimensional crack tip analysis can be used to analyze the elastic behavior of flat cracks in anisotropic elastic bodies. As is shown in the main text, under the action of external uniform stresses, the crack may deform into a thin oblate ellipsoid.

The crack is situated in the x–y plane of a coordinate system, defined by the crack, with respect to which the body is characterized by elastic constants M_{ijkl}. The crack is so oriented in the x–y plane such that points $(\bar{x}, \bar{y})$ on its edge may be parameterized by the angle ϕ:

$$(\bar{x}, \bar{y}) = (a\cos\phi, b\sin\phi)$$

where a and b are respectively the semi-major and semi-minor ellipse axes. Each point s on the edge defines an edge coordinate system $(x_1', x_2', x_3') = (x', y', z')$ where the x'-axis is normal to the edge, the z'-axis is tangent to it, and $y' = z$. The intersection of the x'–z' plane with the flat crack is a 2-D crack, whose near-tip displacements and stresses can be described in terms of the above theory. The crack tip displacements u_i in (x_i') are given in Refs. [10, 11]:

$$u_i' = \mp\frac{\sigma}{E}2\sqrt{\left(\frac{2}{\pi}a\xi\right)}Im(p_{ij}B_j) \tag{A1}$$

using notation established in these references.

The quantities β'_j introduced in eqns (16) and (17) of the main text are related in the text to magnitudes of the crack face displacement. Combining (A1) with these leads to

$$\beta'_j = \frac{-2}{\sqrt{\pi}} Im(p_{jk}B_k) \tag{A2}$$

where the p_{jk} must be evaluated by means of the elastic constants defined with respect to the edge coordinates (x'_i) and are defined in Refs. [10, 11].

It is often convenient to obtain a relation between the β'_i and the k_i. To this end, separate the various field quantities, which are all complex in general, into their real and imaginary parts. In particular

$$p_{jk} = s_{jk} + it_{jk} \tag{A3}$$

$$(N_{jk})^{-1} = U_{jk} + iV_{jk} \tag{A4}$$

where, in Refs. [10, 11], it is shown that

$$k_i = 2N_{ij}B_j\sqrt{a}. \tag{A5}$$

Combining (A3)–(A5) with (A2) leads to

$$k_k = -\sqrt{\pi}\, Q_{jk}\beta'_k \tag{A6}$$

where

$$(Q_{ik})^{-1} = s_{ij}V_{jk} + t_{ij}U_{jk} \tag{A7}$$

and is entirely known.

When (A6) is substituted into the expression for the J-integral of 2-D fracture mechanics (given by eqn (33) of Ref. [10]), it becomes

$$J = -\frac{\sigma^2 a}{E}\frac{\pi}{2} Q_{jk}\beta'_j\beta'_k. \tag{A8}$$

The symmetry of (Q_{ij})

When a two-dimensional Griffith crack is considered, the appropriate matrix Q_{ij} is symmetric. The proof of this follows by means of an application of the Betti–Rayleigh Theorem similar to that presented in Appendix 1 and will not be given here. For a semi-infinite crack, it then follows that Q_{ij} is symmetric because the relations defining the Q_{ij} are independent of crack geometry. They depend upon the asymptotic forms for the stresses and strains close to the crack tip, and these forms are the same for semi-infinite and Griffith cracks.